

## FOURIER TRANSFORM OF THE DERIVATIVE FUNCTION:

The Fourier transform of the function  $u(x, t)$  is given by  $F\{u(x, t)\} = \int_{-\infty}^{\infty} u e^{isx} dx$ ; Then

$$\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx} dx = \left[ e^{isx} \frac{\partial u}{\partial x} - is e^{isx} u \right]_{-\infty}^{\infty} - (is)^2 \int_{-\infty}^{\infty} u e^{isx} dx.$$

If  $u$  and  $\frac{du}{dx}$  tends to zero, as  $x$  tends to  $\pm\infty$ , then

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2 F(u) \quad \text{In general}$$

$$F\left[\frac{\partial^n u}{\partial x^n}\right] = (-is)^n F[u].$$

Provided by the first  $(n-1)$  derivative vanish  $x \rightarrow \pm\infty$

## INVERSE LAPLACE TRANSFORM BY METHOD OF RESIDUES:

Let the Laplace Transform of  $f(x)$  is  $\bar{F}(s)$ , so that  $\bar{F}(s) = \int_0^{\infty} f(t) e^{-st} dt$

multiplied by both sides  $e^{xs}$  and integrating with resp. to  $s$  within limit  $a - i\infty$  to  $a + i\infty$ .

$$e^{xs} \bar{F}(s) ds = \int_{a-i\infty}^{a+i\infty} e^{xs} \int_0^{\infty} f(t) e^{-st} dt ds$$

Put  $s = a - i\eta$ , so that  $ds = -i d\eta$ .

$$= \int_{+\infty}^{-\infty} e^{x(a-i\eta)} \int_0^{\infty} f(t) \cdot e^{(a-i\eta)t} dt \cdot (-i d\eta)$$

$$i e^{ax} \int_{-r}^r e^{ixu} \int_0^u (e^{-at} f(t)) e^{iht} dt du.$$

$$= i e^{ax} \int_{-r}^r e^{ixu} \int_{-\infty}^{\infty} \phi(t) e^{iht} dt du.$$

$$\text{Where } \phi(t) = \begin{cases} e^{-at} f(t) & \text{For } t > 0 \\ 0 & \text{For } t \leq 0. \end{cases}$$

Proceeding to limit as  $r \rightarrow \infty$ , we get

$$\int_{a-i\infty}^{a+i\infty} e^{xs} \bar{F}(s) ds = i e^{ax} \int_{-\infty}^{\infty} e^{-ixu} \int_{-\infty}^{\infty} \phi(t) e^{iht} dt du$$

$$= i e^{ax} \cdot 2\pi \phi(x) \quad \left\{ \text{By Prop. of F.T.} \right.$$

$$= 2\pi i e^{ax} e^{ax} f(x) \quad \text{For } x > 0$$

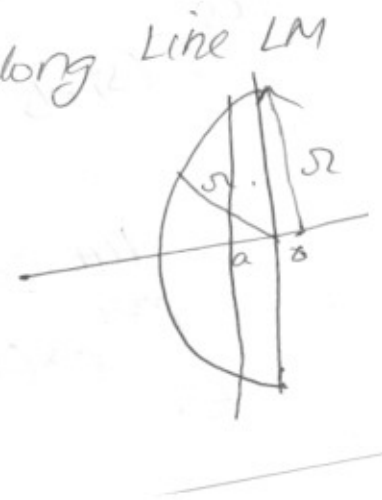
$$\text{Hence } f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{xs} \bar{F}(s) ds \quad (x > 0)$$

Which is called Complex Inversion formula.

It provides a direct means for obtaining the inverse Laplace transform of given function.

The integration is performed along line LM parallel to the imaginary axis

The complex plane  $s = \alpha + i\beta$  such that the singularities of  $F(s)$  lies to left



Let us take a contour  $C$  which is composed  
the line  $LM$  and semicircle  $C'$  (i.e.  $MNU$ )

From (ii)

$$\int_{LM} e^{xs} \bar{F}(s) ds = \frac{1}{2\pi i} \int_C e^{xs} \bar{F}(s) ds - \frac{1}{2\pi i} \int_{C'} e^{xs} \bar{F}(s) ds$$

The integral over  $C'$  tends to zero as  $\sigma \rightarrow \infty$

If +ve constant  $A$  and  $R$  can be so found that

$|F(s)| \leq A \sigma^{-k}$  for every point and  $C'$ , then

$$\lim_{\sigma \rightarrow \infty} \frac{1}{2\pi i} \int_{C'} e^{xs} \bar{F}(s) ds = 0$$

Therefore  $f(x) = \frac{1}{2\pi i} \lim_{\sigma \rightarrow \infty} \int_C e^{xs} \bar{F}(s) ds$

$$\lim_{\sigma \rightarrow \infty} \frac{1}{2\pi i} \int_C e^{xs} \bar{F}(s) ds$$

= Sum of Residues of  $e^{xs} \bar{F}(s)$  at Poles of  $F(s)$

Question  $\rightarrow$  Evaluate  $\mathcal{L}^{-1} \left[ \frac{1}{(s-1)(s^2+1)} \right]$  by Residue method

Since  $\left| \frac{1}{(s-1)(s^2+1)} \right| \sim \left| \frac{1}{s^3} \right|$ ; For  $|s| \rightarrow \infty$ ,  $\therefore$

$$\left[ \frac{1}{(s-1)(s^2+1)} \right] = \text{Sum of Res.} \left[ \frac{e^{xs}}{(s-1)(s^2+1)} \right] \text{ at Poles } s=1, \pm i$$

$$\text{At } \left\{ \text{Res} \right\}_{s=1} = \lim_{s \rightarrow 1} \left[ \frac{(s-1)e^{xs}}{(s-1)(s^2+1)} \right] = \frac{e^x}{2}$$

$$\text{Res} \Big|_{s=i} = \lim_{s \rightarrow i} \left[ \frac{(s-1)e^{xs}}{(s-1)(s^2+1)} \right] = \frac{e^{ix}}{(i-1)(i+1)} = -\frac{e^{ix}}{2(1+i)}$$

By changing  $i$  to  $-i$ , we get

$$\text{Res} \Big|_{s=-i} = -\frac{1}{2} \frac{e^{-ix}}{1-i}$$

$$\mathcal{L}^{-1} \left[ \frac{1}{(s-1)(s^2+1)} \right] = \frac{e^x}{2} - \frac{1}{2} \left[ \frac{e^{ix}}{1+i} + \frac{e^{-ix}}{1-i} \right]$$

$$= \frac{e^x}{2} - \frac{1}{2} \left[ \frac{(1-i)e^{ix} + (1+i)e^{-ix}}{2} \right]$$

$$= \frac{e^x}{2} - \frac{1}{2} \left[ \frac{e^{ix} + e^{-ix}}{2} - i \left( \frac{e^{ix} - e^{-ix}}{2} \right) \right]$$

$$= \frac{1}{2} \left[ e^x - \sin x - \cos x \right]$$

$$= 1 - \frac{2}{\pi} \cdot \frac{\pi}{2} \operatorname{erfc} \left( \frac{c}{2\sqrt{x}} \right).$$

Where follow the Result.

$$\therefore \text{we know that } \int_0^{\infty} e^{-t^2} \cos 2mt \, dt = \frac{1}{2} \sqrt{\pi} \cdot e^{-m^2}$$

Integrating both sides w.r.t.  $m$  from 0 to  $\frac{c}{2\sqrt{x}}$

$$\int_0^{\infty} e^{-t^2} \left| \frac{\sin 2mt}{2t} \right|_{\frac{c}{2\sqrt{x}}} dt = \frac{1}{2} \sqrt{\pi} \int_{\frac{c}{2\sqrt{x}}}^{\infty} e^{-m^2} dm$$

$$\int e^{-t^2} \frac{\sin \frac{2t}{\sqrt{x}}}{t} dt = \frac{\sqrt{\pi}}{2} \operatorname{erfc} \left( \frac{c}{2\sqrt{x}} \right)$$

$$\frac{\pi}{2} \operatorname{erfc} \left( \frac{c}{2\sqrt{x}} \right)$$

As  $\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$ .